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GEOMETRIC STRUCTURE AND MOVING FRAMES FOR THE GENERALIZED TODA LATTICES

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1. INTRODUCTION. The Toda Lattice appeared originally [1] as a mechanical system of particles on the line governed by a certain type of nearest neighbor interaction. With the work of Flaschka [2], the relation with Lie group theory, particularly earlier work by Arnold [3] on the rotating rigid body and its generalizations, came into the foreground; this relation has been extensively developed since, most notably in recent work by Kostant; Olshanetsky and Perelomov; Khazdan, Kostant and Sternberg; and others. In these approaches, the generalizations of the Toda Lattice are developed in terms of the natural symplectic structure on the cotangent bundle of a Lie group.

Now, our present-day notion of "symplectic structure" has its roots in Elie Cartan's book *"Lecons sur les invariant integraux."* The central notion here was that of *Cauchy characteristic* of one or more differential forms. A more definitive version of this concept was given in Cartan's later book *"Les systemes differentiablles exterieures et leurs applications geometriques."* I have set myself the task of developing Cartan's beautiful ideas in the context of contemporary mechanics and physics [8]. In this paper I want to indicate how the work cited above (and some of my own [5]) may be viewed in a Cartanian framework. In addition to the obvious advantage of geometric unification, for its own sake, I believe that certain models with interesting

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) properties might turn up later when this analysis is pushed further.

First, let us review Cartan's ideas, using the standard notations of differential geometry on manifolds [ ]. Let M be a manifold. $D(M)$ denotes the graded associative algebra of differential forms on M. d denotes exterior derivative. An *exterior differential system*, E, is an ideal of $D(M)$ which is closed under d. Given such an E, the vector fields V on M such that

$$i(V)(E) \subset E \tag{1.1}$$

are said to be *Cauchy characteristic* for E. ($i(V)$ denotes the operation of contraction with respect to the vector field V.) The set of all these vector fields defines a foliation for M. (We shall suppose that the foliation is non-singular in the sense that its dimension is constant at each point of M.) If the foliation is regular, i.e. if a quotient map $\pi: M \to M'$ exists whose fibers are the leaves of the foliation and such that M' is a manifold, then the system E lives on M', in the sense that there is an exterior differential system E' on M' such that:

$$E \text{ is generated by } \pi^*(E') .$$

Many applied problems involve determining something about the Cauchy characteristic foliation and the quotient space M'. In problems deriving from mechanics and the calculus of variations, E is generated by a 2-differential form ω such that:

$$d\omega = 0 .$$

In this case, it is readily seen that there is a 2-differential form ω' on M' such that:

$$d\omega' = 0 \text{ and } \omega = \pi^*(\omega') .$$

E' is generated by ω'. Since ω' has no characteristics vectors, it defines M' as a symplectic manifold. Thus we see that Cartan's approach suggests a different insight than that of other recent work on geometrical mechanics; it is not the manifold on which the equations of motion are initially defined that admits the symplectic structure, but the set of all trajectories. Of course, two forms ω and ω_1 could have the same Cauchy characteristic foliation, thus providing two symplectic structures on the trajectories.

One obtains the more traditional equations of mechanics and the calculus of variations by choosing certain local canonical forms for the 2-differential form ω. Typically, these involve natural coordinate systems on the tangent and cotangent bundles to configuration manifolds Q. However, another feature of Cartan's work is what he called the *method of the moving frame*, i.e. the choice of bases of 1-differential forms which are not the differentials of coordinates but which are better adapted to expressing the natural geometric properties of the situation. In this paper I will essentially be adapting this "moving frame" approach to the study of the generalized Toda Lattices. I will start off by assuming that ω has a certain form in terms of certain moving frames for M. Certain general equations will be obtained. In order to understand these equations, I will then specialize to the situation of the work cited above (where M is a submanifold of the cotangent bundle to a Lie group). I hope to present an analysis of other situations in a later work.

2. CAUCHY CHARACTERISTICS OF CLOSED 2-DIFFERENTIAL FORMS.

First, we shall review certain differential-geometric fundamentals [8]. Let M be a manifold, and let ω be a closed 2-differential form on M. If $v \in T(M)$ is a tangent vector to M at a point $p \in M$, the inner product or contraction of ω by v is denoted as $i(v)(\omega)$; it is a 1-covector at p, i.e. an element of the dual space to the tangent space to M at p. Similarly, if V is a vector field on M, $i(V)(\omega)$ is defined as a 1-differential form on M.

DEFINITION. *A tangent vector* $v \in T(M)$ *is said to be Cauchy characteristic for* ω *if* $i(V)(\omega) = 0$. *Similarly, a tangent vector field* V *is said to be Cauchy characteristic if* $i(V)(\omega) = 0$.

In this paper we shall work with a special choice of the manifold M and the closed 2-differential form. Namely, suppose that we are given the following data:

A vector space X and a manifold Y, such that $M = X \times Y$.

An absolute parallelism on Y defined by a basis θ^a, $1 \le a, b \le m$, of 1-differential forms on Y.

A basis x_i, $1 \le i, j \le n$,

of the linear functions on X.

Adopt the summation convention on the indices given above. In addition, suppose that $m > n$; introduce the following additional indices and the summation convention on these indices:

$$n + 1 \le u, \quad v \le m.$$

Let f^a_{bc} be the *structure functions* of the absolute parallelism, ie., the functions on Y such that:

$$d\theta^a = f^a_{bc}\theta^b \wedge \theta^c .$$

Let G be the automorphism group of the absolute parallelism, i.e. the group of diffeomorphism $g: Y \to Y$ such that:

$$g^*(\theta^a) = \theta^a .$$

It is known that G is a Lie group and that it acts simply on Y, i.e. the orbits of G can be identified with G itself. We shall suppose that the orbit space

$$Z = G \backslash Y$$

is a manifold and that the quotient map $Y \to Z$ is a submanifold map. The structure functions f^c_{ab} are constant on the orbits of G, hence are pull-backs under the quotient map of functions on Z. We shall make no notational distinction between these functions.

Now, set:

$$\omega = d(x_i\theta^i) .$$

Let us now compute the Cauchy characteristic vectors of ω, using the relations given above.

$$\omega = dx_i \wedge \theta^i + x_i d\theta^i$$

Our job is to put the differential form ω into its algebraic canonical form. To do this, note that

$$\omega = (dx_i - x_j f_{ai}{}^j \theta^a) \wedge \theta^i$$
$$+ x_i f^i_{uv}\, \theta^u \wedge \theta^v.$$

Set:

$$\alpha_i = dx_i - x_j f^j_{ai} \theta^a$$
$$\omega' = x_i f^i_{uv}\, \theta^u \wedge \theta^v .$$

Then, we have the definitive formula:

$$\omega = \alpha_i \wedge \theta^i + \omega' , \qquad (2.1)$$

where the differential forms on the right hand side of 2.1 are linearly independent. Thus we have proved the following result:

THEOREM 2.1. *The Cauchy characteristic vector fields* V *of* ω *satisfy the following equation:*

$$0 = i(V)(\alpha_i) = i(V)(\theta^i)$$
$$= i(V)(\omega') \qquad (2.2)$$

COROLLARY. *The dimension of the Cauchy characteristic tangent vectors to* ω *is equal to the dimension of the Cauchy characteristic tangent vectors to* ω'.

We can now work out the equations for the Cauchy characteristic vector field V defined by relation (2.2) in more detail. First, let us work with the second relation on the right hand side of (2.2). Suppose that we impose the following relations:

$$\theta^u(V) = h^u \qquad (2.3)$$

where the h^u are functions of the x's and the f's. They must then satisfy the following condition:

$$x_i f^i_{uv} b^u = 0 \tag{2.4}$$

With this choice for these functions, we see that V is completely determined by the first part of relations 2.2:

$$\theta^i(V) = 0 ,$$
$$V(x_i) = x_j f^j_{ui}\, \theta^u(V)$$
$$= x_j f^j_{ui}\, h^u \tag{2.5}$$

Here is an important geometric property of the Cauchy characteristic vector fields of this form which is proved by the relations described above:

THEOREM 2.2. *Let* V *be a Cauchy characteristic vector field of* ω *given by relations* 2.3-.5. *Let* $\phi: M \to X \times Z$ *be the map which sends the point* $(x,y) \in X \times Y = M$ *into* (x,z), *where* z *is the orbit of* G *acting on* Y *which contains the point* y. *Then, the vector field* V *projects into* $X \times Z$, *i.e. there is a vector field* V' *on* $X \times Z$ *such that* ϕ *sends each orbit curve of* V *into an orbit curve of* V'.

In practice, we often start off with V' and construct V. Notice that it is essentially this construction which defines the symplectic structure for the orbit curves of V'; the situation is simplest in case Z reduces to a point, i.e. in case the f's are constants. This means that Y is equal to the Lie group G itself, with G acting by left translation. In this case, we shall see that the equations for the orbit curves of V' are the differential equations for the Toda Lattices and their generalizations.

3. SPECIALIZATION TO THE ABSOLUTE PARALLELISM DEFINED BY THE LEFT INVARIANT DIFFERENTIAL FORMS ON A LIE GROUP. Let us now apply this Theorem 2.1 to the case that the θ^a are the left-invariant Cartan-Maurer forms for a Lie group G, and the x_a

are the dual linear coordinates for the dual vector space $\mathcal{G}^d$ of the Lie algebra of G. Consider X as $\mathcal{J} \times G$, where $\mathcal{J}$ is the linear subspace of $\mathcal{G}^d$ defined by the relations $x_u = 0$. Let $\mathcal{J}'$ be the orthogonal complement of $\mathcal{J}$ in $\mathcal{G}$.

THEOREM 3.1. *At a point* x *of* $\mathcal{J}$, *the dimension of the Cauchy characteristic vectors of the 2-differential form* ω' *is equal to the dimension of the characteristic subspace of the skew-symmetric bilinear form*

$$(J_1,J_2) \to x([J_1,J_2])$$

on $\mathcal{J}'$. *An element* $J_1 \in \mathcal{J}'$ *is Cauchy characteristic for the form* ω' *at* x *iff the following condition is satisfied:*

$$\operatorname{coAd} J_1(x) \in \mathcal{J} \tag{3.1}$$

REMARK. coAd means the dual of the adjoint representation of the Lie algebra $\mathcal{G}$, i.e.

$$\operatorname{coAd}(A)(x)(B) = -x([A,B])$$

$$\text{for } x \in \mathcal{G};\quad A,B \in \mathcal{G}.$$

4. FLASCHKA VECTOR FIELDS ON VECTOR SPACES. The conditions found for Cauchy characteristic vector fields in previous sections are sufficiently interesting and important that it is worth our while to pause and make some general definitions. Let $\mathcal{G}$ be a real Lie algebra. Let $\mathcal{G}^d$ be its dual space. Let X be a linear subspace of $\mathcal{G}^d$. Let X' be the orthogonal complement of X in $\mathcal{G}$, i.e. the set of elements $A \in \mathcal{G}$ such the $X(A) = 0$.

DEFINITION. *A Flaschka map for the vector space* X *is a map* $F: X \to X'$ *such that the following condition is satisfied:*

$$\operatorname{coAd}(F(x))(x) \in X \tag{4.1}$$

for all $x \in X$.

With condition (4.1) satisifed, we can define a vector field V' on X considered as a manifold. Since X is a vector space, a vector field is just a map $X \to X$. Let us then set:

$$V'(x) = \mathrm{coAd}(F(x))(x) \tag{4.2}$$

for all $x \in X$.

Condition (4.1) of course is precisely that which guarantees that V' defined by formula (4.2) is indeed a well-defined vector field on X. We shall call V' a *Flaschka vector field*, since Flaschka's work on the Toda Lattice fits into this framework very naturally.

The orbit curves of the vector fields V' are then the solutions $t \to x(t)$ of the following differential equations:

$$dx/dt = \mathrm{coAd}(F(x(t))(x(t)) \tag{4.3}$$

Conversely, if we start off with the nonlinear differential equations 4.3 (which would be the normal thing to do) we would have the following properties:

THEOREM 4.1. *Consider the system of ordinary differential equations defined by relations* 4.3, *where* F *is a Flaschka map, as defined above. Construct the manifold* M *as* $X \times G$, *and construct the 2-differential form* ω *on* M *as in the previous section. Then, the solution curves of equation* 4.3 *are the projection in* X *of Cauchy characteristic curves of* ω. *In particular, this imposes in a natural way a symplectic structure on the space of solutions of* 4.3.

So far we have been working with an arbitrary Lie algebra $\mathcal{G}$. For a reductive Lie algebra, i.e. one for which $\mathcal{G}$ and $\mathcal{G}^d$ are naturally isomorphic, the formulas can be readily recast so as to be closer to those in the applied mathematics literature.

5. THE FLASCHKA MAPS AND VECTOR FIELDS FOR REDUCTIVE LIE ALGEBRAS. Let us now make the assumption that there is a non-degenerate symmetric bilinear form $\mathcal{B}$ on the (finite dimensional)

Lie algebra $\mathcal{G}$ which is invariant under the adjoint representa- of $\mathcal{G}$ on itself. $\mathcal{B}$ sets up an isomorphism between $\mathcal{G}$ and its dual space $\mathcal{G}^d$. Let $\mathcal{J}$ be a linear subspace of $\mathcal{G}^d$; under this identification of $\mathcal{G}^d$ with $\mathcal{G}$, $\mathcal{J}$ is identified with a linear subspace of $\mathcal{G}$. The orthogonal subspace that we denoted as $\mathcal{J}'$ is then identified with the orthogonal complement of $\mathcal{J}$ with respect to the form $\mathcal{B}$, i.e.

$$\mathcal{J}' = \{A \in \mathcal{J} : \mathcal{B}(A,\mathcal{J}) = 0\}$$

A Flaschka map is then a map

$$B:\mathcal{J} \to \mathcal{J}'$$

such that the vector field

$$V(A) = [B(S),A]$$

is tangent to $\mathcal{J}$.

Here are some ways of constructing such maps.

6. EULER-ARNOLD VECTOR FIELDS ON LIE ALGEBRAS. In this section we shall review certain methods by means of which differential equations may be defined which have some of the properties suggested by the classical rotating rigid body and the recent work on Toda Lattices. These differential equations are essentially defined by means of certain types of vector fields on Lie algebras.

Let $\mathcal{G}$ be a Lie algebra. Since $\mathcal{G}$ is itself a vector space, a vector field (in the sense of manifold theory) is a map $V:\mathcal{G} \to \mathcal{G}$. The orbits or integral curves of such a vector field are the curves $t \to A(t)$ in $\mathcal{G}$ such that

$$dA/dt = V(A(t)) \; . \tag{6.1}$$

Such a vector field will be said to be of *Euler-Arnold type* if it is of the following form

$$V(A) = [B(A),A] \tag{6.2}$$

where B is some map from $\mathcal{G}$ to $\mathcal{G}$.

In [5] I have shown how the construction of such a B and V given in the original Flaschka work on the Toda Lattice may be described in terms of certain "graded" algebra structures on Lie algebras. Here is a more systematic study of this material.

7. JACOBI TRIPLES OF LIE ALGEBRAS. Let $\mathscr{G}$ be a real Lie algebra. A triple $(\mathscr{G}^+, \mathscr{G}^0, \mathscr{G}^-)$ of linear subspaces of $\mathscr{G}$ is said to defined a *Jacobi triple* if the following conditions are satisfied:

$$\begin{aligned} [\mathscr{G}^0, \mathscr{G}^0] &\subset \mathscr{G}^0 \\ [\mathscr{G}^0, \mathscr{G}^{+,-}] &\subset \mathscr{G}^{+,-} \\ [\mathscr{G}^+, \mathscr{G}^-] &\subset \mathscr{G}^0 \ . \end{aligned} \tag{7.1}$$

The linear subspace

$$\mathscr{J} = \mathscr{G}^+ + \mathscr{G}^- + \mathscr{G}^0$$

spanned by these three subspaces is called the *Jacobi subspace* of $\mathscr{G}$ associated with the Jacobi triple.

EXAMPLE. Jacobi Matrices. Let V be a vector space and let $V_1, \dots, V_n$ be linear subspaces of V. Let $\mathscr{G}$ be the Lie algebra (under commutator) of linear maps $V \to V$. Set:

$$\mathscr{G}^0 = \{A \in \mathscr{G} : A(V_i) \subset V_i, \quad i = 1, \dots, n\}$$

$$\mathscr{G}^{+,-} = \{A \in \mathscr{G} : A(V_i) \subset V_{i+,-1}, \quad i = 1, \dots, n\}$$

(Thus $\mathscr{G}^{+,-}$ are the shift up and shift down operators.) It is obvious that the commutation relations 7.1 are satisfied. If the $V_1, \dots, V_n$ are one dimensional linearly independent subspaces which span V, and if a basis is chosen for V consisting of vectors from these subspaces, it is clear that the operators in $\mathscr{J}$ are represented by classical Jacobi matrices, i.e. $n \times n$ matrices with non-zero entries only on the diagonal, sub, and super diagonal lines. It is the point of view developed in [5] that the corresponding gradation of $\mathcal{G}$ gives rise to the Toda Lattice phenomenon.

8. SIMPLE ROOT SYSTEMS FOR SIMPLE LIE ALGEBRAS AND JACOBI TRIPLES. It is known that the Toda Lattice models are closely related to the algebraic properties of Lie algebras, particularly the properties of the simple root systems of semisimple Lie algebra. In this section I will show how the simple root systems generate Jacobi triples.

Let $\mathcal{G}$ be a finite dimensional simple Lie algebra. For the moment, assume that the field of scalars is the complex numbers. We shall return to the case of the real numbers later on. It will of course be assumed that the reader knows basic semisimple Lie algebra theory.

Let $\mathcal{C}$ be a Cartan subalgebra of $\mathcal{G}$. Ad $\mathcal{C}$ acting in $\mathcal{G}$ is then completely reducible. The non-zero eigenvalues of Ad $\mathcal{C}$, considered as linear forms on $\mathcal{C}$, are called the roots of the Lie algebra. Let $r = \dim \mathcal{C}$. (r is the rank of the Lie algebra $\mathcal{G}$). A set $\lambda_1,\dots,\lambda_r$ of roots is said to define a simple root system if the following conditions are satisfied:

> Each root λ can be written as a linear combination of the $\lambda_1,\dots,\lambda_r$ with coefficients which are integers, and which are simultaneously all non-negative or non-positive. $-\lambda_1,\dots,-\lambda_r$ are roots.

That such simple root systems exist, and serve to determine the isomorphism class of the Lie algebra, is a well-known fact of Lie algebra theory.

Fix such a simple root system. Let (A_i), $i = 1,\dots,r$ be a collection of root elelemtns, i.e. vectors of $\mathcal{G}$ such that

$$[A,A_i] = \lambda_i(A)A_i (\text{no summation})$$

for all $A \in \mathcal{G}$.

(It is known from Lie algebra theory that there is, up to a constant multiple, just one such root vector.)

Since $\lambda_i - \lambda_j$ is not a root, for $i \neq j$, we have:

$$[A_i, A_j] = 0 \quad \text{for } i \neq j .$$

Similarly, let A_{-i} be a chain of root vectors for the root $-\lambda_i$. We then have

$$[A_{-i}, A_{-j}] = 0 \quad i \neq j .$$

Also,

$$[A_i, A_{-i}] \in \mathcal{G}$$

Thus, if we let $\mathcal{G}^+$ $(\mathcal{G}_0^-)$ be the linear subspace spanned by the $A_i(A_{-i})$, and let $\mathcal{G}^0$ be $\mathcal{G}$, we see that $(\mathcal{G}^0, \mathcal{G}^+, \mathcal{G}^-)$ forms a Jacobi triple.

So far $\mathcal{G}$ has been a complex Lie algebra. We can form a real subalgebra $\mathcal{G}'$ as the Lie subalgebra generated by the A_i, A_{-i}. $(\mathcal{G}^0, \mathcal{G}^+, \mathcal{G}^-)$ forms a Jacobi triple in $\mathcal{G}'$.

9. EULER-ARNOLD VECTOR FIELDS WHICH ARE TANGENT TO THE JACOBI SUBSPACES. Let $\mathcal{G}$ be a real Lie algebra and let $(\mathcal{G}^+, \mathcal{G}^-, \mathcal{G}^0)$ be a Jacobi triple of linear subspaces. Let $\mathcal{J}$ be the associated Jacobi subspace of $\mathcal{G}$. Let $B: \mathcal{G} \to \mathcal{G}$ be a map, and set:

$$V(A) = [B(A), A] \quad \text{for } A \in \mathcal{G}$$

Consider V as a vector field on $\mathcal{G}$. We ask:

When is V tangent to the linear subspace $\mathcal{J}$?

To answer this question, suppose that $B = B^+ + B^0 + B^-$, where

$$B^{+,-,0}(\mathcal{J}) \subset \mathcal{G}^{+,-,0}$$

If $A = A^+ + A^- + A^0 \in \mathcal{J}$, then

$$V(A) = [B^+(A) + B^0(A) + B^-(A),\ A^+ + A^0 + A^0]$$

Let us now suppose that the following conditions are satisfied:

$$[B^+(A), A^+] = 0$$
$$[B^-(A), A^-] = 0 \qquad (9.1)$$

THEOREM 9.1. *If conditions* 9.1 *are satisfied, then the Euler-Arnold vector field* $V(A) = [B(A),A]$ *is tangent to the Jacobi subspace* $\mathcal{J}$.

Proof. Condition 9.1 implies that $V(\mathcal{J}) \subset \mathcal{J}$, which is the condition that $\mathcal{J}$ be tangent to $\mathcal{J}$.

10. JACOBI TRIPLES DEFINED BY AUTOMORPHISMS OF LIE ALGEBRAS. Following a suggestion by Victor Kac (private communication), we shall now show how Jacobi triples may be defined by automorphisms of Lie algebras. Let $\mathcal{G}$ be a finite dimensional Lie algebra, with the complex numbers as field of scalars. Let $\sigma:\mathcal{G} \to \mathcal{G}$ be an automorphism of this Lie algebra. Let $\lambda \in C$ be an eigenvalue of σ. Set:

$$\mathcal{G}^0 = \{A \in \mathcal{G} : \sigma(A) = A\}$$

$$\mathcal{G}^+ = \{A \in \mathcal{G} : \sigma(A) = \lambda A\}$$

$$\mathcal{G}^- = \{A \in \mathcal{G} : \sigma(A) = \lambda^{-1} A\}$$

These three subspaces $(\mathcal{G}^0, \mathcal{G}^+, \mathcal{G}^-)$ then define a Jacobi triple. This way of defining Jacobi triples is especially natural because automorphisms are classifiable if $\mathcal{G}$ is a semisimple Lie algebra. The case where the automorphism is of finite order would be of particular interest since Kac has classified them. (See Section 5, Chapter X of [20]. It is also shown in this reference how such automorphisms are related to graded structures on Lie algebras.)

As an illustration let us construct the automorphism which gives rise to the classical Jacobi matrices. Let V be a finite dimensional complex vector space. Let

$$V = V_1 + \dots + V_n$$

be a direct sum decomposition of V as a direct sum of linear subspaces. Let λ be a primitive n-th root of unity, i.e.

$$\lambda^n = 1, \quad \text{but} \quad \lambda^j \neq 1 \quad \text{for} \quad j = 2,\dots,n-1 .$$

Set

$$\sigma(v) = \lambda^j v \quad \text{for} \quad v \in V_j, \quad j = 1,\ldots,n \ .$$

Let $\mathscr{G}$ be $L(V,V)$, the Lie algebra of all linear maps: $V \to V$. σ defines a linear map (also denoted as σ) of $\mathscr{G} \to \mathscr{G}$.

$$\sigma(A) = \sigma A \sigma^{-1} \quad \text{for} \quad A \in \mathscr{G}$$

Set

$$\mathscr{G}^0 = \{A \in \mathscr{G} : \sigma(A) = A\}$$

$$\mathscr{G}^+ = \{A \in \mathscr{G} : \sigma(A) = \lambda A\}$$

$$\mathscr{G}^- = \{A \in \mathscr{G} : \sigma(A) = \lambda^{-1} A\} \ .$$

We see that

$$\mathscr{G}^0(V_j) \subset V_j \quad \text{for} \quad j = 1,\ldots,n \ .$$

$$\mathscr{G}^+(V_j) \subset V_{j+1}$$

$$\mathscr{G}^-(V_j) \subset V_{j-1} \ .$$

These relations imply that $\mathscr{G} = \mathscr{G}^0 + \mathscr{G}^+ + \mathscr{G}^-$ is a linear subspace of $\mathscr{G}$ which is represented by Jacobi matrices in the classical sense, when a basis is chosen for V consisting of elements in the subspaces V_j.

11. FLASCHKA MAPS CONSTRUCTED FROM JACOBI TRIPLES. Suppose now that $\mathscr{G}$ is a reductive Lie algebra. Notice that the Euler-Arnold objects differ from the Flaschkan objects only in that the values of the former type of map do not necessarily lie in the appropriate orthogonal complement. In this section we shall see what sort of compatibility condition between the graded structures and the AdG-invariant symmetric bilinear form $\mathscr{B}: \mathscr{G} \times \mathscr{G} \to R$ must be imposed in order to assure that the Flaschka conditions are to be satisfied.

Suppose that $(\mathscr{J}^-, \mathscr{J}^+, \mathscr{J}^0)$ is a Jacobi triple of linear subspaces of the Lie algebra $\mathscr{G}$. In addition, let us suppose that $T: \mathscr{G} \to \mathscr{G}$ is a linear map such that the following condition is satisfied:

$$T[A_1,A_2]) = [T(A_2),T(A_1)] \tag{11.1}$$

$$\mathscr{B}(T(A_1),T(A_2)) = \mathscr{B}(A_1,A_2) \tag{11.2}$$

for $A_1,A_2 \in \mathscr{G}$.

$$T(T(A)) = A \quad \text{for } A \in \mathscr{G} \tag{11.3}$$

REMARK. Notice that, if $\mathscr{G}$ is a Lie algebra of matrices, then $T(A)$ = transpose of A as a matrix, will have these properties.

Let us now suppose that the Jacobi triple, the T-operation and the form $\mathscr{B}$ satisfy the following conditions:

$$T(\mathscr{J}^{\pm}) = \mathscr{J}^{+} \tag{11.4}$$

$$T(\mathscr{J}^{0}) = \mathscr{J}^{0} \tag{11.5}$$

$$\mathscr{B}(\mathscr{J}^{\pm},\mathscr{J}^{\pm}) = 0 \tag{11.6}$$

$$\mathscr{B}(\mathscr{J}^{\pm},\mathscr{J}^{0}) = 0 \tag{11.7}$$

Let us now define the linear subspace $\mathscr{J}$ of $\mathscr{G}$ as the set of all elements of $\mathscr{G}$ of the following form:

$$J = J^{-} + J^{0} + T(J^{-}). \tag{11.8}$$

where J^{-}, J^{0} are arbitrary elements of $\mathscr{J}^{-},\mathscr{J}^{0}$. Define the linear map $B\colon \mathscr{J} \to \mathscr{G}$ by the following formula:

$$B(J) = J^{-} - T(J^{-}) \tag{11.9}$$

THEOREM 11.1. *The map defined by formula 11.9 has the Flaschka property, i.e. the following conditions are satisfied:*

$$[B(J),J] \in \mathscr{J}, \quad \textit{and}$$

$$B(J) \in \mathscr{J}'.$$

for $J \in \mathscr{J}$, *and with* $\mathscr{J}'$ *the orthogonal complement of with respect to the form* $\mathscr{B}$.

The *proof* of these statements is now an easy consequence of our assumptions.

12. CONCLUSIONS. We have introduced a different geometric way of constructing the ordinary differential equations models which generalize the Toda Lattice, and which might be candidates for "integrable" or "partially integrable" systems. Instead of starting off with a Lie group given, as in the previous work, we have begun at one stage more general, with an "absolute parallelism." In the case where this parallelism does come from a Lie group, we have codified some of the algebraic conditions which seem to be involved. There is also potential for extension to the field-theoretic situations by means of generalizations to infinite dimensional manifolds, groups and Lie algebras.

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